# On the Arithmetic of Polynomial Semidomains

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# The guiding question

## Problem

Let $S$ be a semidomain. Which factorization properties of $S$ survive after adjoining a polynomial variable?

The extensions in this talk are the polynomial semidomain $S[x]$ and the Laurent polynomial semidomain $S[x^{\pm 1}]$.

The properties under consideration are:

atomicity, ACCP, BFS, FFS, UFS, LFS.

# Why semidomains?

A semidomain is a subsemiring of an integral domain.

This single class contains two familiar sources of examples.

- Integral domains, where additive inverses are present.
- Positive semirings, such as subsemirings of $\mathbb{R}_{\geq 0}$.

The polynomial semidomain $\mathbb{N}_0[x]$ already behaves differently from $\mathbb{Z}[x]$. For instance, $\mathbb{N}_0$ has unique factorization, but $\mathbb{N}_0[x]$ does not.

# The factorization hierarchy

Assume that the multiplicative monoid is atomic. The usual implications are:

$$\begin{array}{ccccccc} \text{UFS} & \Longrightarrow & \text{HFS} & & & & \\ \Downarrow & & \Downarrow & & & & \\ \text{FFS} & \Longrightarrow & \text{BFS} & \Longrightarrow & \text{ACCP} & \Longrightarrow & \text{atomic.} \end{array}$$

The talk asks how much of this hierarchy is stable under

$$S \longmapsto S[x] \qquad \text{and} \qquad S \longmapsto S[x^{\pm 1}].$$

# A first example to keep in mind

The semidomain $\mathbb{N}_0$ is a UFS.

However, the following element of $\mathbb{N}_0[x]$ has two distinct factorizations:

$$x^5 + x^4 + x^3 + x^2 + x + 1 = (x + 1)(x^4 + x^2 + 1) = (x^2 + x + 1)(x^3 + 1).$$

The factors appearing in this display are irreducible in $\mathbb{N}_0[x]$. Thus unique factorization does not ascend from $\mathbb{N}_0$ to $\mathbb{N}_0[x]$.

# Notation for factorizations

Let $M$ be a commutative cancellative monoid.

- The group of units is denoted by $\mathsf{U}(M)$.
- The reduced monoid is $M_{\mathrm{red}}$.
- The atoms of $M$ form the set $\mathcal{A}(M)$.

When $M$ is atomic, the free monoid of factorizations is

$$\mathsf{Z}(M) := \mathcal{F}\big(\mathcal{A}(M_{\mathrm{red}})\big).$$

For $b \in M$, the set of factorizations and the set of lengths are defined by

$$\mathsf{Z}_M(b) := \pi^{-1}\big(b\,\mathsf{U}(M)\big), \qquad \mathsf{L}_M(b) := \{\, |z| : z \in \mathsf{Z}_M(b)\,\}.$$

# Semidomains and their polynomial extensions

Let $S$ be a semidomain.

- The multiplicative monoid is $S^* := S \setminus \{0\}$.
- The group of multiplicative units is $S^\times$.
- The atoms of $S$ mean the atoms of $S^*$.

The Laurent extension has a useful unit group:

$$S[x^{\pm 1}]^\times = \{\, ux^n \colon u \in S^\times \text{ and } n \in \mathbb{Z} \,\}.$$

This makes Laurent polynomials close enough to polynomials for factorization questions, while still allowing powers of $x$ to be inverted.

# The ascent picture

The main results split into three types.

| property | ascent to $S[x]$ and $S[x^{\pm 1}]$ | conclusion |
|---|---|---|
| ACCP | yes | equivalent to the property for $S$ |
| BFS | yes | equivalent to the property for $S$ |
| FFS | yes | equivalent to the property for $S$ |
| atomicity | conditional | controlled by maximal common divisors |
| UFS/LFS | restrictive | forces the domain case |

The rest of the talk explains this table and the examples behind it.

# Atomicity needs a coefficient condition

A nonzero polynomial $f \in S[x]$ is *indecomposable* if it cannot be written as a product of two nonconstant polynomials in $S[x]$.

A common divisor $d$ of a finite set $A \subseteq S^*$ is a *maximal common divisor* if each common divisor of

$$\{\, a/d \colon a \in A \,\}$$

is a unit.

The set of maximal common divisors is denoted by $\mathrm{mcd}(A)$.

# The atomicity theorem

**Theorem**

For a semidomain $S$, the following statements are equivalent.

1. The semidomain $S$ is atomic, and for each indecomposable polynomial
$$f = \sum_{i=0}^{n} c_i x^i \in S[x]^*,$$
the set $\operatorname{mcd}(c_0, \ldots, c_n)$ is nonempty.
2. The polynomial semidomain $S[x]$ is atomic.
3. The Laurent polynomial semidomain $S[x^{\pm 1}]$ is atomic.

Thus atomicity does not ascend for free. The obstruction is visible in the coefficients of indecomposable polynomials.

# Proof idea for atomicity

The key step is the implication from the coefficient condition to atomicity of $S[x]$.

Suppose that $S[x]$ is not atomic. Choose a minimum-degree nonunit polynomial $f$ that does not factor into irreducibles.

- Minimality forces $f$ to be indecomposable.
- A maximal common divisor of the coefficients gives a factorization $f = cg$.
- The polynomial $g$ is irreducible because its coefficients have no nonunit common divisor.
- The constant $c$ would then fail to factor in $S$, contradicting atomicity of $S$.

# MCD-semidomains behave well

**Proposition**

For a semidomain $S$, the following statements are equivalent.

1. The semidomain $S$ is an MCD-semidomain.
2. The polynomial semidomain $S[x]$ is an MCD-semidomain.
3. The Laurent polynomial semidomain $S[x^{\pm 1}]$ is an MCD-semidomain.

Combining this proposition with the atomicity theorem yields an important consequence.

If $S$ is an atomic MCD-semidomain, then $S[x]$ and $S[x^{\pm 1}]$ are atomic.

# Power series are different

The good behavior of $S[x]$ and $S[x^{\pm 1}]$ does not extend to formal power series.

**Example**

The semidomain $\mathbb{N}_0[[x]]$ is not atomic, even though $\mathbb{N}_0$ is a UFS.

The central element is the geometric series

$$f := \sum_{i=0}^{\infty} x^i.$$

The proof shows that every nontrivial factorization of $f$ can be refined. This prevents $f$ from having an atomic factorization.

# ACCP ascends

**Theorem**

For a semidomain $S$, the following statements are equivalent.

1. The semidomain $S$ satisfies ACCP.
2. The polynomial semidomain $S[x]$ satisfies ACCP.
3. The Laurent polynomial semidomain $S[x^{\pm 1}]$ satisfies ACCP.

The proof follows the classical strategy. A chain in $S[x]$ is controlled by two pieces of data: degree and leading coefficient.

# Proof idea for ACCP

Consider an ascending chain of principal ideals in $S[x]$:

$$f_1 S[x] \subseteq f_2 S[x] \subseteq f_3 S[x] \subseteq \cdots .$$

The divisibility $f_{n+1} \mid f_n$ gives two constraints.

- The degrees form a nonincreasing sequence.
- The leading coefficients form an ascending chain of principal ideals in $S$.

Once both pieces stabilize, the original chain stabilizes. Laurent polynomials are reduced to polynomial ones by multiplying by a power of $x$.

# Atomicity does not imply ACCP

The implication

$$\text{ACCP} \implies \text{atomic}$$

is not reversible.

## Example

Take $r \in \mathbb{Q} \cap (0,1)$ with numerator at least 2, and let

$$S_r := \langle\, r^n \colon n \in \mathbb{N}_0 \,\rangle.$$

The positive semiring generated additively by

$$\{\, e^s \colon s \in S_r \,\}$$

is atomic but does not satisfy ACCP.

This example is not an integral domain.

# Bounded factorizations

A monoid $M$ is a BFM if and only if it has a length function.

A length function is a map $\ell \colon M \to \mathbb{N}_0$ such that:

1. the equality $\ell(u) = 0$ holds exactly for $u \in \mathrm{U}(M)$;
2. the inequality $\ell(bc) \geq \ell(b) + \ell(c)$ holds for all $b, c \in M$.

**Theorem**

For a semidomain $S$, the following statements are equivalent: $S$ is a BFS, $S[x]$ is a BFS, and $S[x^{\pm 1}]$ is a BFS.

# Proof idea for BFS

Start with a length function $\ell\colon S^* \to \mathbb{N}_0$.

For $f \in S[x]^*$, use the leading coefficient $c(f)$ and the degree:

$$\ell_x(f) := \ell(c(f)) + \deg f.$$

This defines a length function on $S[x]^*$.

For Laurent polynomials, shift $f$ into $S[x]$ by using its order:

$$\overline{\ell}(f) := \ell\left(f/x^{\operatorname{ord} f}\right).$$

This defines a length function on $S[x^{\pm 1}]^*$.

# Finite factorizations

**Theorem**

For a semidomain $S$, the following statements are equivalent.

1. The semidomain $S$ is an FFS.
2. The polynomial semidomain $S[x]$ is an FFS.
3. The Laurent polynomial semidomain $S[x^{\pm 1}]$ is an FFS.

The proof uses two finiteness sources:

- finitely many possible leading coefficients up to associates in $S$;
- finitely many divisors in an ambient polynomial ring $K[x]$.

# A BFS need not be an FFS

The implication

$$\text{FFS} \Longrightarrow \text{BFS}$$

is not reversible.

**Example**

For $k \in \mathbb{N}$ with $k \geq 2$, the positive semiring

$$S := \mathbb{N}_0 \cup \mathbb{R}_{\geq k}$$

is a BFS but not an FFS.

The reason is that elements near $k+1$ produce infinitely many nonassociate factorizations of $(k+1)^2$.

# Unique factorization fails quickly

The semidomain $\mathbb{N}_0$ is a UFS, but the polynomial semidomain $\mathbb{N}_0[x]$ is not a UFS.

The same polynomial also shows the failure of length-factoriality:

$$x^5 + x^4 + x^3 + x^2 + x + 1 = (x + 1)(x^4 + x^2 + 1) = (x^2 + x + 1)(x^3 + 1).$$

Both factorizations have length 2. They are distinct.

# The obstruction is structural

**Lemma**

Let $S$ be a semidomain that is not an integral domain. The following polynomials are irreducible in $S[x]$:

$$x + 1, \qquad x^2 + x + 1, \qquad x^3 + 1, \qquad x^4 + x^2 + 1.$$

The equality from the previous slide therefore produces distinct equal-length factorizations over every non-domain semidomain.

# Length-factoriality forces domains

**Proposition**

Let $S$ be a semidomain. If $S[x]$ is an LFS, then $S$ is an integral domain.

In other words, if additive inverses are genuinely missing, then the polynomial semidomain cannot be length-factorial.

This explains why unique factorization and length-factoriality behave very differently from ACCP, BFS, and FFS in this setting.

# The final classification

## Theorem

For a semidomain $S$, the following statements are equivalent.

1. The semidomain $S$ is a UFD.
2. The polynomial semidomain $S[x]$ is a UFS.
3. The Laurent polynomial semidomain $S[x^{\pm 1}]$ is a UFS.
4. The semidomain $S$ is an LFD.
5. The polynomial semidomain $S[x]$ is an LFS.
6. The Laurent polynomial semidomain $S[x^{\pm 1}]$ is an LFS.

Thus UFS and LFS ascend precisely in the classical domain situation.

# MCD versus GCD

Maximal common divisors are robust under polynomial extension:

$$S \text{ is MCD} \iff S[x] \text{ is MCD} \iff S[x^{\pm 1}] \text{ is MCD.}$$

Greatest common divisors behave differently.

**Example**

The semidomain $\mathbb{N}_0$ is a UFS and hence a GCD-semidomain. However, $\mathbb{N}_0[x]$ is not a GCD-semidomain.

# Far from half-factoriality

Assume that $S[x]$ is atomic and that $(S, +)$ is reduced.

## Proposition

The polynomial semidomain $S[x]$ has full and infinite elasticity.

The construction uses the polynomial

$$f_{n,k}(x) := (x+n)^n(x^2 - x + 1)(x+1)^k.$$

Its relevant factorizations have lengths $k+1$ and $k+n$, so the ratios

$$\frac{k+n}{k+1}$$

realize every rational number at least 1.

# What survives polynomial extension?

The picture for semidomains is mixed.

- ACCP, BFS, and FFS ascend to $S[x]$ and $S[x^{\pm 1}]$.
- Atomicity ascends under a maximal-common-divisor condition.
- The MCD property ascends and descends.
- The GCD property does not ascend in general.
- UFS and LFS ascend only in the integral-domain case.

# Takeaway

Polynomial semidomains extend the classical arithmetic of polynomial rings, but they keep track of what is lost when additive inverses disappear.

The slogan is:

finiteness properties ascend,      factoriality does not.

Thank you!